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INFLUENCE OF INTERFACIAL MODELS ON A STRESS FIELD NEAR A CRACK TERMINATING AT A BIMATERIAL INTERFACE

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Abstract—Mode III problem for a semi-infinite crack terminating at bimaterial interface is investigated. The interface is modelled by different interfacial contact conditions. Corresponding boundary value problems are reduced to functional-different equations by the Fourier and Mellin integral transforms, and later to systems of singular integral equations with fixed points singularities. Solutions of the problems for various interfacial contact conditions are found. Particularly, the singularity of stresses near the crack tip is calculated. Numerical results of the singularity exponents, stress intensity factors, and jumps of the displacement near the crack tip are presented. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

It is a known fact from the papers by Williams (1959) and Zak and Williams (1963) that an exponent of the stresses singularity is not equal to 0.5 as a rule, when the crack tip is placed at the bimaterial interface. This result is a consequence of the "ideal" interfacial conditions (tractions and displacements are continuous) along the interface. For such values of the singularity exponent the usual Griffith-Irwin fracture criterion cannot be directly applied. To remedy this for the interface crack, many authors following Comninou (1977) assumed a frictionless contact zone along the crack surfaces near crack tip. In such an approach, an oscillation of displacements near the crack tip is absent and the singularity exponent is already equal to 0.5. (For intensive literature on this topic see Rice (1988) and Comninou (1990).) In order to also obtain the same singularity for arbitrary crack location Atkinson (1979), the author (1985), Erdogan *et al.* (1991), Ozturk and Erdogan (1995) and many others presupposed that there was a special thin (but not equal to zero) transition layer, into which mechanical properties are varied between the materials. The third way to investigate fracture process near the "ideal" bimaterial interface is to use kinked crack approach. He and Hatchinson (1989) have shown that such method makes it possible to answer a question whether the crack will deflect into the interface or penetrate it for various positions of the crack with respect to the interface. However, as Cherepanov (1984), and Geubelle and Knauss (1994) have noted, the fact that the singularity exponent is not equal to 0.5 has a principal character.

On the other hand, such criteria as the critical crack opening criterion (Leonov and Panasyuk (1959); Dugdale (1960); Wells (1961)) or the effective stresses criterion (Novozhylov (1969)) can be directly used for arbitrary stresses field, according to which mechanism of fracture should be taken into account. Note in this connection the non-local stress failure condition by Seweryn and Mróz (1995) which generalized the above mentioned criteria. Consequently, one can consider any stress distribution near the crack tip which depends in an essential way on models of bimaterial interface as well as on the boundary conditions along crack surfaces. It is interesting to note that (as it has been shown by Nazarov (1981)) even if the crack is placed in homogeneous material the singularity exponent can be equal to a certain value between 0 and 0.5 according to the contact boundary conditions along the crack surfaces.

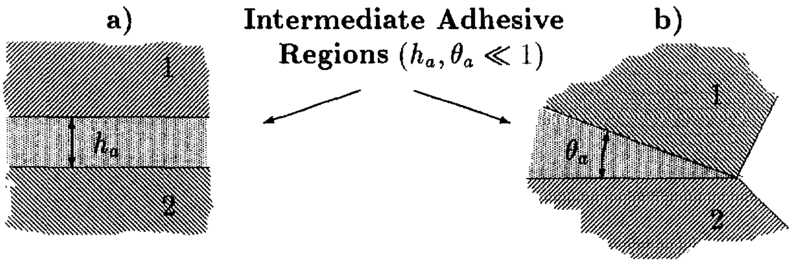

Fig. 1.

Interfacial conditions other than those that are mentioned above are considered in problems of soil and rock mechanics and mechanics of composites for layered structures. Namely, tractions are assumed to be continuous and are proportional to displacements discontinuity: $([u]-\tau(x)\sigma)_{|\Gamma} = 0, [\sigma]_{|\Gamma} = 0$ along the interface Γ (see Linkov and Filippov (1991)). Here $[f]_{|\Gamma}$ denotes a jump of the vector-function f across the interface; $\tau(x)$ is some matrix-function, but x are local coordinates changing along the interface. With theory of elasticity point of view such the contact conditions arise when bodies interact by a thin adhesive elastic region, which is interpreted as a thin shell. So, if 2D problems are considered, and the interact regions are a thin layer, or a thin wedge (as in Figs 1a, b), then the matrix-functions $\tau(x)$ are of the form $\tau(x) = \tau_L$; $\tau(x) = r\tau_W$, respectively, where τ_L, τ_W are constant diagonal matrices, but (r, θ) are the local coordinates connected with the wedge tip.

The main regularities of interaction between the crack approaching to such "nonideal" interfaces from the matrix have been investigated by the author (1994) for Mode III. In this paper, an influence of "nonideal" interfacial conditions on the stresses singularity near the crack tip terminating at the interface is investigated. We consider Mode III for the domain which is such simple as it is possible—homogeneous elastic half-plane and two wedges (Fig. 2). Such geometry presents the semi-infinite crack terminating at the bimaterial interface. We assume that the adhesive intermediate region consists of a thin layer and two thin wedges. Then corresponding "nonideal" interfacial conditions are rewritten in the form: $[\sigma_{\theta z}]_{|\Gamma_\pm} = 0$, $([u_z]-(\tau+r\tau_\pm)\sigma_{\theta z})_{|\Gamma_\pm} = 0$. Here $\tau = h_a/\mu_a$, $\tau_\pm = \theta_a^\pm/\mu_a^\pm$ $(\tau, \tau_\pm \ll 1)$, but μ_a, $\mu_a^\pm$ are the shear moduli of the thin layer and the wedges, respectively. These equations generalize the "ideal" contact conditions $(\tau, \tau_\pm = 0)$, and can be considered independently on the particular model of the thin interconnecting region. Then the parameters τ, $\tau_\pm$ can be interpreted as measures of flexibility of the adhesive.

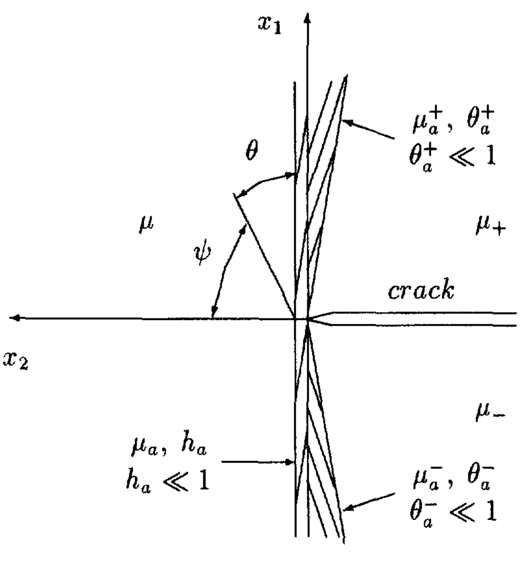

Fig. 2. Geometry of the problem under consideration.

In the second and third sections the problem is formulated and is reduced to a system of functional-difference equations. The solution of this system is found in the fourth section for the case $\tau = 0$ (when the intermediate region is represented by thin adhesive wedges only). The general case $\tau > 0$ is considered in the last section. In the Appendix the system of functional-difference equations is reduced to a system of singular integral equations with fixed points singularities, which is investigated in proper functional spaces.

2. PROBLEM FORMULATION

Let us consider an infinite domain consisting of the half-plane Ω and two wedges $\Omega_\pm$:

$$\Omega = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 > 0\}, \quad \Omega_\pm = \{(r,\theta) : r \in \mathbb{R}_+, \pm(\theta + \pi/2) \in (0, \pi/2)\}.$$

We shall seek the function $u(x_1, x_2)$ which satisfies the Laplace equation inside each of the regions Ω, $\Omega_\pm$. Exterior boundary conditions along crack surfaces $\Gamma_0^\pm = \{(r,\theta) : r \in \mathbb{R}_+, \theta = -\pi/2 \pm 0\}$ are defined in the form:

$$\mu_\pm \frac{1}{r}\frac{\partial}{\partial \theta} u_{\pm}\Big|_{\Gamma_0^\pm} = \pm q_\pm(r), \quad \text{where} \quad \int_0^\infty [q_+(r) + q_-(r)]\, \mathrm{d}r = 0. \tag{1}$$

The last equation for the functions $q_\pm(r)$ in (1) is the usual condition for solvability of the Neumann problem (mechanical sense of this condition is also evident). Assume also that the functions $q_\pm \in C_0^\infty(\mathbb{R}_+)$. Then any singularities of the solutions will be connected with the interior properties of the problems only.

Along the interior boundaries $\Gamma_\pm = \{(x_1, x_2) : \pm x_1 \in \mathbb{R}_-, x_2 = 0\}$ (between the half-plane Ω and the wedges $\Omega_\pm$) the conditions hold the constants $\tau, \tau_\pm \geqslant 0$

$$\begin{cases} \left(u - u_\pm - \mu\tau \dfrac{\partial}{\partial x_2} u \mp \mu_\pm \tau_\pm \dfrac{\partial}{\partial \theta} u_\pm\right)\Big|_{\Gamma_\pm} = 0, \\[2ex] \dfrac{\partial}{\partial x_2}(\mu u - \mu_\pm u_\pm)\Big|_{\Gamma_\pm} = 0. \end{cases} \tag{2}$$

We shall look for regular solutions of the problems (1)–(2) in a class of functions LW such that $u \in LW$ if the following relations are held true:

1. $u|_G \in \mathbf{C}^2(G)$;
2. $u(x_1, x_2) = O(r^{-\gamma_\infty})$, $\quad r = \sqrt{x_1^2 + x_2^2} \to \infty$, $\quad (x_1, x_2) \in G$,
3. $u(x_1, x_2) = u_* + O(r^{\gamma_0})$, $\quad r \to 0$, $\quad (x_1, x_2) \in \Omega$,

$$u(x_1, x_2) = u_* + v_\pm + O(r^{\gamma_0}), \quad r \to 0, \quad (x_1, x_2) \in \Omega_\pm. \tag{3}$$

Here G denotes corresponding regions (Ω or $\Omega_\pm$) and γ_0, γ_∞ ($0 < \gamma_0 \leqslant 1, 0 < \gamma_\infty < 1$) are certain constants which will be found through solving the problem.

3. REDUCTION TO A SYSTEM OF FUNCTIONAL EQUATIONS

Applying the fourier and Mellin transforms

$$\bar{u}(\lambda, x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\lambda x_1} u(x_1, x_2)\, \mathrm{d}x_1, \quad \tilde{u}(s, \theta) = \int_0^\infty u(r, \theta) r^{s-1}\, \mathrm{d}r, \tag{4}$$

to the Laplace equation in the respective regions, obtain

$$\bar{u}(\lambda, x_2) = C(\lambda)\, \mathrm{e}^{-\lambda|x_2|}, \quad \tilde{u}_\pm(s,\theta) = A_\pm(s)\cos(s\theta) + B_\pm(s)\sin(s\theta). \tag{5}$$

From *a priori* estimates (3) of a solution of the problem belonging to the class LW, and the properties of the Mellin transform, it follows that the functions $A_\pm(s)$, $B_\pm(s)$ are analytical in the strip $0 < \mathscr{R}s < \gamma_\infty$, and have simple poles in the point $s = 0$, in general. Moreover, these functions can be analytically extended on the strip $-\gamma_0 < \mathscr{R}s < \gamma_\infty$. It will be convenient to us to represent the function $C(\lambda)$ as a sum of the odd and even components: $C(\lambda) = C_+(\lambda) + C_-(\lambda)$, then from (3) and the properties of the Fourier transform, it follows

$$C_\pm(\lambda) = \begin{cases} O(\lambda^{-1-\gamma_0}), & \lambda \to \infty, \\ O(\lambda^{\gamma_\infty - 1}), & \lambda \to 0, \end{cases} \quad C_{\pm|_{\mathbb{R}_+}} \in L^{1,\alpha,\beta}(\mathbb{R}_+), \tag{6}$$

with the parameters $\alpha > 1 - \gamma_\infty$, $\beta < \gamma_0 + 1$. Here $L^{p,\alpha,\beta}(\mathbb{R}_+)$ is the Banach space of summable with the weight functions of the norm:

$$\|u\|^{p,\alpha,\beta} = \left(\int_0^\infty |u(\xi)|^p \rho^p_{\alpha,\beta}(\xi)\, \mathrm{d}\xi/\xi \right)^{1/p}, \quad \rho_{\alpha,\beta}(\xi) = \begin{cases} \xi^\alpha, & 0 < \xi < 1 \\ \xi^\beta, & 1 \leqslant \xi < \infty \end{cases}. \tag{7}$$

The exterior boundary conditions (1) are rewritten in the form:

$$A_\pm(s)\sin(\pi s/2) + B_\pm(s)\cos(\pi s/2) = \pm(\mu_\pm s)^{-1}\tilde{q}_\pm(s+1). \tag{8}$$

Moreover, we can write balance conditions for each of the regions, taking into account (4):

$$\lim_{\lambda\to 0} \bar{u}'_{x_2}(\lambda, 0) = 0, \quad \lim_{s\to 0} \tilde{u}'_\theta(s,\theta)_{|\Gamma_\pm} = \pm(\mu_\pm)^{-1}\tilde{g}(1). \tag{9}$$

Now we fit together the Fourier and the Mellin transformations along the common boundaries $\Gamma_\pm$ (conditions (2)) in a similar manner as in Mishuris and Olesiak (1995). For this aim represent the first equations as follows:

$$\left(1 - \mu\tau \frac{\partial}{\partial x_2}\right) \int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda x_1} \bar{u}(\lambda, x_2)\, \mathrm{d}\lambda_{|\Gamma_\pm} = \left(u_\pm \pm \mu_\pm \tau_\pm \frac{\partial}{\partial\theta} u_\pm\right)_{|\Gamma_\pm},$$

or

$$\int_0^\infty (1 + \mu\tau|\lambda|)[C_+(\lambda)\cos\lambda r \mp iC_-(\lambda)\sin\lambda r]\, \mathrm{d}\lambda = \left(u_\pm \pm \mu_\pm\tau_\pm \frac{\partial}{\partial\theta} u_\pm\right)_{\Gamma_\pm}.$$

Further, applying the Mellin transform to both parts of the equations, and taking into account the identities (see Gradshteyn and Ryzhik (1965)):

$$\int_0^\infty r^{s-1} \begin{Bmatrix} \sin r\lambda \\ \cos r\lambda \end{Bmatrix} \mathrm{d}r = \Gamma(s) \begin{Bmatrix} \sin(\pi s/2) \\ \cos(\pi s/2) \end{Bmatrix} \lambda^{-s}, \begin{Bmatrix} -1 < \mathscr{R}s < 1 \\ 0 < \mathscr{R}s < 1 \end{Bmatrix}, \tag{10}$$

we obtain the equations:

$$\begin{aligned} &2\Gamma(s)[(\hat{C}_+(s) + \mu\tau\hat{C}_+(s-1))\cos(\pi s/2) - i(\hat{C}_-(s) + \mu\tau\hat{C}_-(s-1))\sin(\pi s/2)] \\ &\quad = A_+(s) + \mu_+\tau_+ sB_+(s), \\ &2\Gamma(s)[(\hat{C}_+(s) + \mu\tau\hat{C}_+(s-1))\cos(\pi s/2) + i(\hat{C}_-(s) + \mu\tau\hat{C}_-(s-1))\sin(\pi s/2)] \\ &\quad = A_-(s)\cos(\pi s) - B_-(s)\sin(\pi s) - \mu_-\tau_- s[B_-(s)\cos(\pi s) + A_-(s)\sin(\pi s)]. \end{aligned} \tag{11}$$

Here, we define $\hat{f}(s) \equiv \tilde{f}(1-s)$. The second equations from (2) are analogously rewritten in the form:

$$2\Gamma(s)(\hat{C}_{-}(s)\sin(\pi s/2)+i\hat{C}_{+}(s)\cos(\pi s/2)) = \frac{\mu_+}{\mu}B_+(s),$$

$$2\Gamma(s)(\hat{C}_{+}(s)\sin(\pi s/2)-i\hat{C}_{-}(s)\cos(\pi s/2)) = -\frac{\mu_-}{\mu}(B_-(s)\cos(\pi s)+(A_-(s)\sin(\pi s)). \quad (12)$$

As it follows from (6) and the properties of the Mellin transform the functions $\hat{C}_{\pm}(s)$ are analytical in the strips $-\gamma_0 < \mathscr{R}s < \gamma_\infty$. Hence, the functions $\hat{C}_{\pm}(s)$, $\hat{C}_{\pm}(s-1)$ have at least the common strip of analyticity $1-\gamma_0 < \mathscr{R}s < \gamma_\infty$. Of course, we assume in this case that $\gamma_\infty+\gamma_0 > 1$. So, the eqns (11), (12) hold in the strip $1-\gamma_0 < \mathscr{R}s < \gamma_\infty$, in general. Eliminating from (8), (11), (12) the functions $A_{\pm}(s)$, $B_{\pm}(s)$, through algebra the following system of functional equations is obtained:

$$\mu\tau\hat{\mathbf{C}}(s-1)+\Phi(s)\hat{\mathbf{C}}(s) = F(s), \quad (13)$$

where $\mathbf{C}(s) = (C_+(s), iC_-(s))^{\mathrm{T}}$, $\Phi(s) = \Phi_1(s)+\Phi_2(s)$, and the matrix-functions $\Phi_1(s)$, $\Phi_2(s)$ and the vector-function $F(s)$ are defined like this:

$$\Phi_1(s) = \begin{pmatrix} 1+m_+ & m_-\operatorname{ctg}(\pi s/2) \\ -m_-\operatorname{ctg}(\pi s/2) & 1-m_+\operatorname{ctg}^2(\pi s/2) \end{pmatrix}, \quad t_{\pm} = \frac{\tau_+ \pm \tau_-}{2},$$

$$\Phi_2(s) = \begin{pmatrix} -\mu t_+ s\operatorname{tg}(\pi s/2) & -\mu t_- s \\ \mu t_- s & \mu t_+ s\operatorname{ctg}(\pi s/2) \end{pmatrix}, \quad m_{\pm} = \frac{\mu(\mu_- \pm \mu_+)}{2\mu_+\mu_-},$$

$$F(s) = \frac{1}{2\mu_+\mu_-\Gamma(s+1)\sin(\pi s)}\begin{pmatrix} \mu_+\tilde{q}_-(s+1)+\mu_-\tilde{q}_+(s+1) \\ [\mu_+\tilde{q}_-(s+1)-\mu_-\tilde{q}_+(s+1)]\operatorname{ctg}(\pi s/2) \end{pmatrix}. \quad (14)$$

Note that the first condition (9) has been justified in view to (5), (6). The next conditions (9) are rewritten in the form $2\mu i\hat{C}_-(0) = \pm\tilde{g}_{\pm}(1)$, where only one of them

$$2\mu i\hat{C}_-(0) = \tilde{g}_+(1) \quad (15)$$

should be taken into account in view to the condition from (1).

4. SOLVING THE PROBLEM IN THE CASE $\tau = 0$

It means that geometry of the intermediate adhesive region between the wedges and the half-plane is modelled by two thin wedges only. In this case, the system of the eqns (13) is correctly defined at least in the strip $0 < \mathscr{R}s < \gamma_\infty$, because the term $\hat{\mathbf{C}}(s-1)$ is absent in (13). It is easy to note that the matrix-function $\Phi(s)$ is invertible into a strip containing the imaginary axis. In fact, the determinant of the matrix $\Phi(s)$

$$\det\Phi(s) = 1+m_+-(m_+^2+m_+-m_-^2)\operatorname{ctg}^2(\pi s/2)-\mu^2 s^2(t_+^2-t_-^2)$$
$$+s\mu t_+(\operatorname{ctg}(\pi s/2)-\operatorname{tg}(\pi s/2))+2\mu s(m_+t_+-m_-t_-)\operatorname{ctg}(\pi s/2) \quad (16)$$

is the even function of the argument s, and the function $\det\Phi(i\xi)$ is not equal to zero for all $\xi \in \mathbb{R}$. Moreover, the zero ω_1 ($\mathscr{R}\omega_1 > 0$) of the function $\det\Phi(s)$, which is closest to the imaginary axis is simple and real. Thus, the matrix-function $\Phi^{-1}(s)$ is analytical in the strip $-\omega_1 < \mathscr{R}s < \omega_1$, and it does not increase along any line parallel to the imaginary axis. Then the eqn (13) is solved in a closed form as

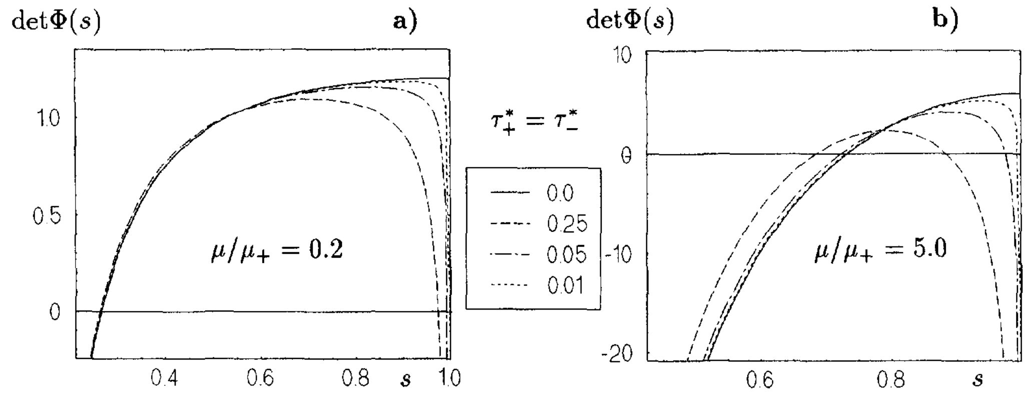

Fig. 3. The graphs of the function $\det\Phi(s)$ on the interval $(0, 1)$ in the case $\mu_+ = \mu_-$, $\tau_+ = \tau_-$.

$$\hat{C}(s) = \hat{H}(s) \equiv \Phi^{-1}(s)F(s), \quad H(\lambda) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Phi^{-1}(s)F(s)\lambda^{s-1}\,\mathrm{d}s, \tag{17}$$

and it can be directly verified $2\mu\hat{h}_2(0) = \tilde{g}_+(1)$, hence the condition (9) holds.

It can be easily shown that $H \in L_2^{p,\alpha,\beta}(\mathbb{R}_+)$ for any $p \in [1, \infty)$, $\alpha > 1 - \omega_1$, $\beta < 1 + \omega_1$. Because of $q_\pm \in C_0^\infty$, we can more exactly investigate the asymptotics of the functions $C_\pm(\lambda)$. To this aim, we need information about the next poles of the vector-function $\hat{H}(s)$. Note that there exist two simple real zeros of the function $\det\Phi(s)$ in the interval $(0, 1)$, in general. And, if we deal with the "ideal" contact condition ($\tau_\pm = 0$) only, there will exist a unique zero of the function $\det\Phi(s)$ in the interval $(0, 1)$. Moreover, the next (the third in the case $\tau_+ + \tau_- > 0$, or the second for $\tau_\pm = 0$) zero of this function is placed in half-plane $\mathscr{R}s > 1$.

When mechanical parameters are symmetrical ($\mu_- = \mu_-, \tau_+ = \tau_-$) with respect to the OX_2-axis, typical graphs of the function $\det\Phi(s)$ are presented in Fig. 3a, b.

Here continuous lines correspond to the "ideal" ($\tau_\pm^* = 0$) contact, but discontinuous lines correspond to "non-ideal" contact with respective values of the parameters $\tau_\pm^* = 0.01, 0.05, 0.25$. Further on we will be using more convenient parameters $\tau^* = \mu\tau$, $\tau_\pm^* = \mu_\pm\tau_\pm$, which have dimensions as length. In general, the parameters τ^*, $\tau_\pm^*$ are not small in comparison with the parameters τ, $\tau_\pm$.

For this case ($\tau_+ = \tau_-$), the graphs of the first and the second zeros of the function $\det\Phi(s)$ are presented in Fig. 4 in a logarithmic scale with respect to the parameter $\tau_\pm^*$ for various values of the ratio μ/μ_+. Note that the values of the first zero ω_1 for small magnitudes of $\tau_\pm^*$ ($\tau_\pm^* < 0.1$) are a little different from the values of the unique zero for the "ideal" contact condition ($\tau_\pm^* = 0$). Besides, the values of the second zero ω_2 are an approximate unity for $\tau_\pm^* < 0.005$. The graphs of the zero ω_2 for small $\tau_\pm^*$ are presented in Fig. 4c. As the value of $\tau_\pm^*$ increases, the values of both zeros decrease. Moreover, ω_1, ω_2 tend to zero, when $\tau_\pm^* \to \infty$. The last result is evident, at least for the first zero. In fact, the contact condition for the displacement can be rewritten in the form

$$\sigma_{zx_2} = \frac{1}{r\tau_+}[u].$$

Then, passing to the limit $\tau_+ \to \infty$, this condition will correspond to the homogeneous boundary condition for the traction.

The other important case is, when along the boundary Γ_- "ideal" contact condition holds ($\tau_-^* = 0$), but along the boundary Γ_+, there is the "non-ideal" condition ($\tau_+^* > 0$). The elasticity parameters are assumed to be $\mu_+ = \mu_-$ as above. Then the graphs of the first and the second zeros of the function $\det\Phi(s)$ are presented in Fig. 5 with respect to the parameter τ_+^* for various values of the ratio μ/μ_+.

As it follows from Fig. 5, the tendency for a decrease of both zeros is preserved, but the second zero ω_2 does not tend to zero when $\tau_+^* \to \infty$ for this contact condition.

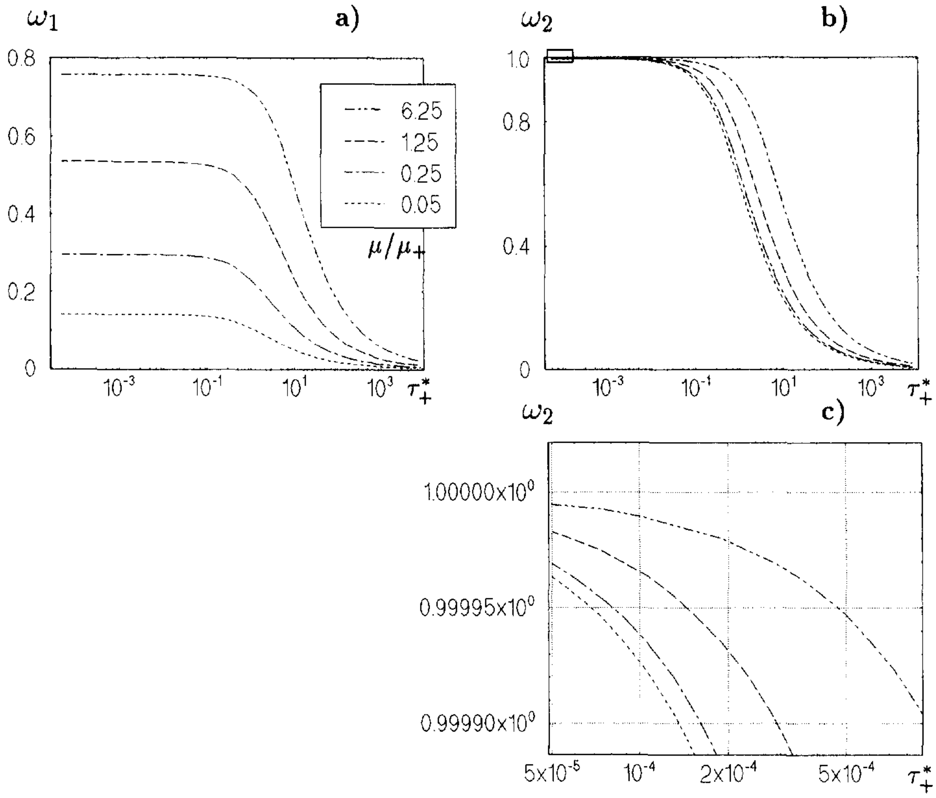

Fig. 4. The graphs of the first zero ω_1 (a) and the second zero ω_2 (b) for the case $\mu_+ = \mu_-$, $\tau_+^* = \tau_-^*$ in a logarithmic scale against τ_+^* for different values of the ratio μ/μ_+; (c) corresponds to the zoomed region shown by rectangular in (b).

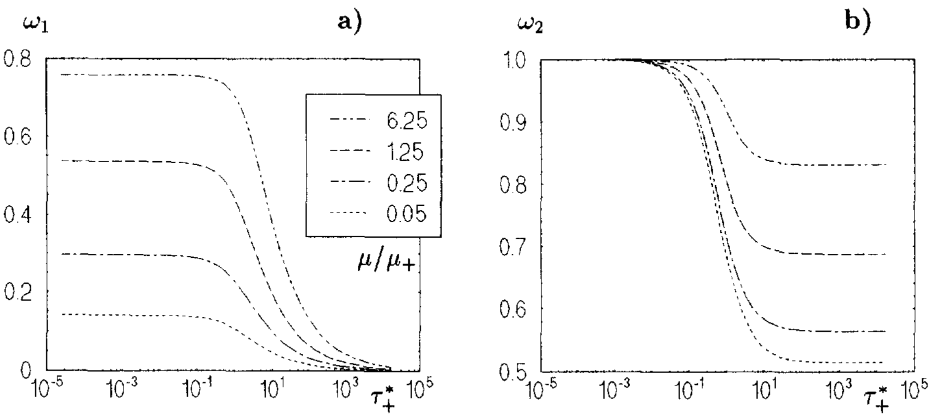

Fig. 5. The graphs of the first zero ω_1 (a) and the second zero ω_2 (b) for the case $\mu_+ = \mu_-$, $\tau_-^* = 0$ in a logarithmic scale against τ_+^* for different values of the ratio μ/μ_+.

So, the asymptotics of the solution $\mathbf{C}(\lambda)$ $(=H(\lambda))$ at infinity, which are needed to find the asymptotics of the problem solution near crack tip, can be written as follows:

$$H(\lambda) = \sum_{j=1}^{2} \begin{pmatrix} c_+^{(j)} \\ c_-^{(j)} \end{pmatrix} \lambda^{-\omega_j - 1} + o(\lambda^{-2-\varepsilon}), \quad \lambda \to \infty, \tag{18}$$

where $c_\pm^{(j)}$ are residues of the respective components of the vector-function $\hat{H}(s)$ in the points $s = -\omega_j$ $(j = 1, 2)$, and $\varepsilon > 0$ is some small value. The behaviour of this vector-function in the neighbourhood of the point $\lambda = 0$ can be obtained in an analogous way: $H(\lambda) = O(\lambda^{\omega_1 - 1})$, $\lambda \to 0$.

Now, we can calculate the problem solution into the region Ω by the inverse Fourier transform:

$$u(x_1, x_2) = 2\int_0^{\infty} [C_+(\lambda)\cos\lambda x_1 - iC_-(\lambda)\sin\lambda x_1]\,e^{-\lambda x_2}\,d\lambda.$$

Using the inverse Mellin transform the last relation can be rewritten as follows ($0 < \delta < \omega_1$):

$$u(x_1, x_2) = \frac{1}{\pi i}\int_{-i\infty+\delta}^{i\infty+\delta} \left\{\hat{C}_+(s)\int_0^{\infty} \lambda^{s-1}\cos\lambda x_1\,e^{-\lambda x_2}\,d\lambda - i\hat{C}_-(s)\int_0^{\infty}\lambda^{s-1}\sin\lambda x_1\,e^{-\lambda x_2}\,d\lambda\right\}ds.$$

Here all transformations are justified in view of (6). Further, using identities similar to (10) ($x_1 = r\cos\theta, x_2 = r\sin\theta, \psi = \pi/2-\theta, \theta\in[0,\pi]$):

$$\int_0^{\infty} \lambda^{s-1}\begin{Bmatrix}\sin x_1\lambda\\ \cos x_1\lambda\end{Bmatrix} e^{-\lambda x_2}\,d\lambda = \Gamma(s)\begin{Bmatrix}\cos s\psi\\ \sin s\psi\end{Bmatrix} r^{-s}, 0 < \mathscr{R}s < 1,$$

we obtain

$$u(x_1, x_2) = \frac{1}{\pi i}\int_{-i\infty+\delta}^{i\infty+\delta} \Gamma(s)\{\hat{C}_+(s)\cos s\psi - i\hat{C}_-(s)\sin s\psi\}r^{-s}\,ds. \tag{19}$$

In order to calculate the solution in the remaining region $\Omega_\pm$ the following relations are needed

$$\tilde{u}_+(s,\theta) = \frac{\tilde{g}_\pm(s+1)}{\mu_\pm s}\,\frac{\cos[s(\theta+\pi/2\mp\pi/2)]}{\sin(\pi s/2)} - \frac{2\mu}{\mu_\pm}\cos(s(\theta+\pi/2))\Gamma(s)[\hat{C}_-(s)\pm i\hat{C}_-(s)\,\mathrm{ctg}(\pi s/2)], \tag{20}$$

which follows from (5), (8) and (12). Let us note that the right-hand side of (20) has a simple pole in the point $s = 0$ only in view to (15).

Finally, the asymptotics of the displacement $u(r,\theta)$ in a local system of coordinates (r,θ) coinciding with the crack tip ($x_1 = r\cos\theta, x_2 = r\sin\theta, \psi = \pi/2-\theta$) can be found by the residue theorem:

$$u(r,\theta) \underset{r\to 0}{=} -\sum_{j=1}^{2}\frac{2}{\omega_j}\Gamma(1-\omega_j)r^{\omega_j}\begin{cases}[c_+^{(j)}\cos\omega_j\psi + c_-^{(j)}\sin\omega_j\psi], & (r,\theta)\in\Omega\\ \dfrac{\mu}{\mu_\pm}\cos(\omega_j(\theta+\pi/2))[c_+^{(j)}\mp c_-^{(j)}\,\mathrm{ctg}\,(\pi\omega_j/2)], & (r,\theta)\in\Omega_\pm\end{cases}$$

$$+u_0 + o(r), r\to 0, u_0 = \frac{1}{\pi(2\mu+\mu_++\mu_-)}\int_0^{\infty}[g_+(r)+g_-(r)]\ln r\,dr, \tag{21}$$

The last relation (21) can be rewritten in the usual form

$$u(r,\theta) = u_0 + \sum_{j=1}^{2}\begin{cases}(\mu\omega_j)^{-1}k_{\mathrm{III}}^{(j)}f^{(j)}(\psi)r^{\omega_j}, & (r,\theta)\in\Omega,\\ (\mu_\pm\omega_j)^{-1}k_{\mathrm{III}}^{(j)}f_\pm^{(j)}(\theta)r^{\omega_j}, & (r,\theta)\in\Omega_\pm\end{cases} + o(r), r\to 0. \tag{22}$$

Here $k_{\mathrm{III}}^{(i)}$ are constants which play a role stress intensity factors (S.I.F.), and they are equal to S.I.F. in the case $\omega_i = 0.5$. In general, these constants and the functions $f^{(j)}(\psi), f_\pm^{(j)}(\theta)$ are defined by the relations

$$k_{\mathrm{III}}^{(j)} = \lim_{s\to-\omega_j}\left[\frac{\mathrm{d}}{\mathrm{d}s}\det\Phi(s)\right]^{-1}\frac{\mu}{\mu_+\mu_-\sin\pi\omega_j}[1-m_+\operatorname{ctg}^2(\pi\omega_j/2)+\mu t_+\omega_j\operatorname{ctg}(\pi\omega_j/2)]$$

$$\times\{\mu_+[1+\Lambda_j\operatorname{ctg}(\pi\omega_j/2)]\tilde{g}_-(1-\omega_j)+\mu_-[1-\Lambda_j\operatorname{ctg}(\pi\omega_j/2)]\tilde{g}_+(1-\omega_j)\},$$

$$f^{(j)}(\psi) = \cos(\psi\omega_j)+\Lambda_j\sin(\psi\omega_j), f_\pm^{(j)}(\theta) = \cos(\omega_j(\theta+\pi/2))[1\mp\Lambda_j\operatorname{ctg}(\pi\omega_j/2)], \quad (23)$$

where

$$\Lambda_j = \frac{\mu t_-\omega_j - m_-\operatorname{ctg}(\pi\omega_j/2)}{1-m_+\operatorname{ctg}^2(\pi\omega_j/2)+\mu t_+\omega_j\operatorname{ctg}(\pi\omega_j/2)}.$$

However, for the symmetrical problem ($\mu_+ = \mu_-, \tau_+ = \tau_-$), the relations (23) cannot be used, because the matrix-function $\Phi(s)$ has diagonal elements $\phi_j(s)$ ($j = 1, 2$) only (see (16)). In this special case, the respective relations for S.I.F. and the functions from (22) are in the forms

$$k_{\mathrm{III}}^{(j)} = \lim_{s\to-\omega_j}\left[\frac{\mathrm{d}}{\mathrm{d}s}\phi_j(s)\right]^{-1}\frac{\mu}{\mu_+\sin\pi\omega_j}\begin{cases}[\tilde{g}_+(1-\omega_1)-\tilde{g}_-(1-\omega_1)]\operatorname{ctg}(\pi\omega_1/2), & j=1,\\ \tilde{g}_-(1-\omega_2)+\tilde{g}_-(1-\omega_2), & j=2,\end{cases}$$

$$f^{(1)}(\psi) = \sin(\psi\omega_1), \quad f^{(2)}(\psi) = \cos(\psi\omega_2),$$

$$f_\pm^{(1)}(\theta) = \mp\cos(\omega_1(\theta+\pi/2))\operatorname{ctg}(\pi\omega_1/2), \quad f_\pm^{(2)}(\theta) = \cos(\omega_2(\theta+\pi/2)), \quad (24)$$

The asymptotics (22) makes it possible to easily obtain the asymptotics of stresses near crack tip.

It is interesting to note that if not only the mechanical parameters $\mu_\pm$, $\tau_\pm$, but the tractions along the crack surfaces will be symmetrical also ($g_+(r) = -g_-(r)$), then the coefficient $k_{\mathrm{III}}^{(2)}$ will be equal to zero in the term corresponding to the second zero ω_2.

In Fig. 6a the graphs of the coefficient $k_\delta^{(1)} = k_{\mathrm{III}}^{(1)}$ are presented against the parameter τ_+^* for the various ratios of the elasticity constants μ/μ_+. The traction is assumed to be the Dirac δ-function on the unity distance from the crack tip. As one can see, the value of $k_\delta^{(1)}$ decreases (and tends to zero!) when the magnitude of the ratio μ/μ_+ increases. This fact shows that the value of generalized stress intensity factor (the parameter in the main singular term of stresses) cannot be unique of the fracture mechanics parameter. A more important value to investigate the process of quasibrittle fracture is $k_{\mathrm{III}}^{(j)}d^{\omega_j}/\omega_j$, where d is a certain small geometrical parameter of a material (e.g. Mishuris and Semenov (1986); Seweryn (1994)). It appears when the effective stresses criterion by Novozhilov (1969), or the critical crack opening criterion by Leonov and Panasyuk (1959), Dugdale (1960) and Wells (1961), or generalized criterion by Seweryn and Mróz (1995) are applied. In Fig. 6b, the graphs of the value $k_\delta^{(1)}/\omega_1$ are presented. As one can see, the value of $k_\delta^{(1)}/\omega_1$ does not

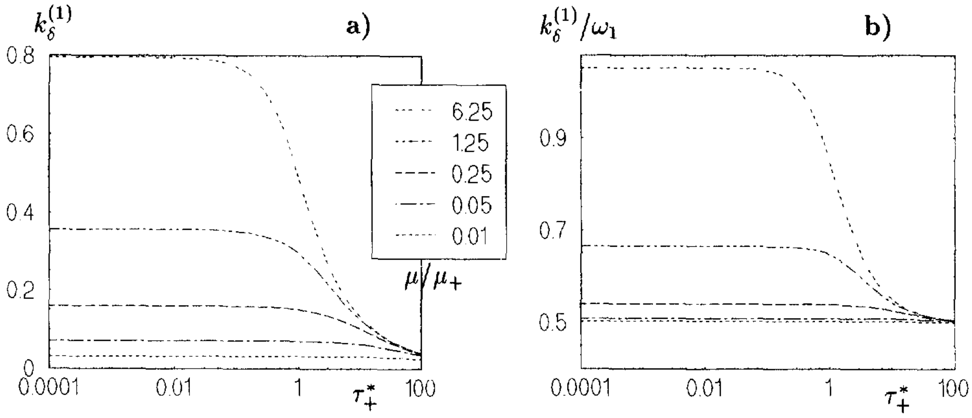

Fig. 6. The graphs of the coefficient $k_\delta^{(1)}$ (a) and the ratio $k_\delta^{(1)}/\omega_1$ (b) for the case $\mu_- = \mu_+$, $\tau_-^* = \tau_+^*$ in a logarithmic scale against τ_+^* for different values of the ratio μ/μ_+.

tend to zero when the magnitude of the ratio μ/μ_+ increases. Moreover, taking into account the behaviour of ω_1 (see Figs 4a, 5a) and the fact that $d \ll 1$, the value $k_\delta^{(1)} d^{\omega_1}/\omega_1$ already increases when the magnitude of the ratio μ/μ_+ increases, which seems to be a natural occurrence.

In conclusion, note that for the case under consideration ($\tau = 0$), all values of the parameters from the class LW have been found $\gamma_0 = \gamma_\infty = \omega_1, v_\pm = 0$ and $u_* = u_0$ (see (21)).

5. SOLVING THE PROBLEM IN A GENERAL CASE ($\tau > 0$)

First of all, note that the system of the functional eqns (13) holds in the whole strip $0 < \mathscr{R}s < \gamma_\infty$, namely, from the *a priori* estimates (see (12)) the vector-function $\hat{\mathbf{C}}(s)$ is analytic in the strip $-\gamma_0 < \mathscr{R}s < \gamma_\infty$, but the vector-function $F(s)$ and the matrix-function $\Phi(s)$ are analytic in the strip $0 < \mathscr{R}s < 1$. Then from (13) it follows that the vector-function $\hat{\mathbf{C}}(s-1)$ should be analytic in the strip $0 < \mathscr{R}s < \gamma_\infty$ and can have a double pole in the point $s = 0$. Hence, in this case ($\tau > 0$) the parameter $\gamma_0 \geqslant 1$, and

$$\mathbf{C} \in L_2^{1,1-\omega_1+\varepsilon,2-\varepsilon}(\mathbb{R}_-), \tag{25}$$

for any $\varepsilon > 0$. Multiply the system of the eqns (13) by the matrix $\Phi^{-1}(s)$

$$\tau^*\Phi^{-1}(s)\hat{\mathbf{C}}(s-1)+\hat{\mathbf{C}}(s) = \hat{H}(s). \tag{26}$$

Then, by investigating the behaviour of the functions $\hat{C}_+(s-1)$, $\hat{C}_-(s-1)$ near the point $s = 0$, and taking into account (15) and (25), one can easy see that only the function $\hat{C}_-(s-1)$ can have a simple pole in this point, consequently

$$\mathbf{C}(\lambda) = \begin{pmatrix} 0 \\ e_- \end{pmatrix}\lambda^{-2}+o(\lambda^{-2}), \quad \lambda \to \infty, \tag{27}$$

where $e_- \in \mathbb{R}$ is some unknown constant, which will be found later, besides

$$\hat{C}_-(0) = \hat{h}_1(0)+\frac{\pi\tau^*}{2[m_+^2+m_+-m_-^2]}\left(m_-e_-+\frac{2m_+}{\pi}\hat{C}_+(-1)\right).$$

Now, we shall consider the system of the eqn (26) with the additional condition (15) for three cases separately, as the matrix-function $\Phi^{-1}(s)$ has special behaviour at infinity into the strip of analyticity.

FIRST CASE: $\tau_\pm^* = 0$, but $\tau^* \neq 0$. This is the situation when the intermediate adhesive layer is the thin layer only, then $\Phi^{-1}(s) = O(1)$, $\det\Phi^{-1}(s) = O(1)$, $|\mathscr{F}s| \to \infty$.

SECOND CASE: $\tau^*, \tau_\pm^* > 0$ ($t_+^2 \neq t_-^2$). It means that the intermediate adhesive region is modelled by the thin layer and the thin wedges along each of the interior boundaries $\Gamma_\pm$: $\Phi^{-1}(s) = O(1/s)$, $\det\Phi^{-1}(s) = O(1/s^2)$, $|\mathscr{F}s| \to \infty$.

THIRD CASE: τ^*, $\tau_+^* > 0$, but $\tau_-^* = 0$ ($t_+ = t_-$). This is the situation when the intermediate adhesive region is modelled by the thin layer and the thin wedge along one of the interior boundaries Γ_- only. Then $\Phi^{-1}(s) = O(1)$, $\det\Phi^{-1}(s) = O(1/s)$, $|\mathscr{F}s| \to \infty$, and the main part of the matrix-function $\Phi^{-1}(s)$ is degenerated at infinity.

In the Appendix the systems (26) with the additional condition (15) in the spaces (25) for all cases are reduced to systems of singular integral equations with fixed point singularities. Solvability of such systems and the convergence of numerical methods are proved.

Particularly, in the first and the second cases when all mechanical parameters are symmetrical with respect to the OX_2-axis ($\mu_+ = \mu_-, \tau_+ = \tau_-$ or $m_- = 0, t_- = 0$) corresponding systems of the integral equations (A8) split on two independent equations, as would be expected. Besides, in the first case, the solution of the first equation is found in a closed form:

$$C_+(\lambda) = \frac{1+m_+}{1+m_+ + \tau^*\lambda} h_1(\lambda),$$

but the next equation can be written as follows:

$$y_2(\lambda) + \frac{2\lambda\tau^*}{\pi[1+m_+ + \tau^*\lambda](1+m_+)} \int_0^\infty \frac{(\lambda/\xi)^{2-\omega_1} - (\lambda/\xi)^{\omega_1}}{[(\lambda/\xi)^2 - 1]\sin \pi\omega_1} y_2(\xi) \frac{d\xi}{\lambda}$$

$$= \frac{(1+m_+)\lambda}{1+m_+ + \tau^*\lambda} \frac{1}{2\pi i} \int_{-i\infty+\delta}^{i\infty+\delta} \hat{h}_2^*(s) \operatorname{ctg}\frac{\pi s}{2} \lambda^{s-1}\, ds, |\delta| < \omega_1.$$

Here $h_2^*(\lambda)$ is the second component of the vector function $H^*(\lambda)$ defined in (A3), but $iC_-(\lambda) = \hat{h}_2(0)e^{-\lambda} + \mathscr{B}_1[y_2](\lambda)$, where operator $\mathscr{B}_1$ is defined in (A9).

Now, we assume that the solution of (26) from the space (25) is already found. Then it is possible to obtain the asymptotics of the problem solution near the crack tip, using the representation (19), (20), and the asymptotics (27) ($x_1 = r\cos\theta, x_2 = r\sin\theta, \psi = \pi/2 - \theta$):

$$u(r,\theta) \underset{r\to 0}{=} u_* + \begin{cases} -2r\{[\hat{C}_-(-1) - e_-\psi]\cos\psi - [a + e_- \ln r]\sin\psi\}, & (r,\theta)\in\Omega \\ v_\pm - \dfrac{\mu}{\mu_\pm \tau^*} v_\pm r \sin\theta, & (r,\theta)\in\Omega_\pm \end{cases} + O(r^{1+\omega_1}),$$

(28)

where

$$a = \lim_{s\to -1} \frac{d}{ds}[(s+1)^2\Gamma(s) i\hat{C}_-(s)] = -e_-[1+\Gamma'(1)] - \lim_{s\to -1}\frac{d}{ds}[(s+1)i\hat{C}_-(s)],$$

$$u_* = u_0 - \frac{\pi\tau^*}{m_+^2 + m_+ - m_-^2}[m_- e_- + 2m_+\pi^{-1}\hat{C}_+(-1)], \quad v_\pm = \tau^*[2\hat{C}_+(-1) \mp \pi e_-].$$

It is interesting to note that only in the region Ω do the stresses have weak singularity (as $\ln r$). Such a singularity arises also (Zwiers *et al.*, 1982) at the free edge of composite laminate, and can play an important role (see Stolarski and Chiang (1989)).

The greatest value of the stresses $\sigma_{\theta z}$ arises beyond the crack line into the region Ω (i.e., when $\psi = 0$ or what is equivalent $\theta = \pi/2$):

$$\max_{\theta\in[0,\pi]} \sigma_{\theta z}(r,\theta) = \sigma_{\theta z}(r,\pi/2) = -2\mu\{a + e_-(1+\ln r)\} + O(r^{\omega_1}), \quad r\to 0.$$

This seems to be a natural occurrence. However, along the interior boundaries $\Gamma_\pm$ between the wedges $\Omega_\pm$ and the half-plane Ω ($\psi = \pm\pi/2$) the tractions are already bounded.

Now we present distributions of the displacement discontinuity $[u](r,0)$ along the interface Γ_+ in the case when all mechanical parameters are symmetric with respect to the Ox_2-axis ($\tau_\pm^* = 0, \mu_+ = \mu_-$). The tractions along crack surfaces are normalised: $g_+(r) = -g_-(r) = \mu_+ e^{-r}$. Besides, we assumed that the value of shear modulus μ_+ is invariant under this consideration, then the value of m_- changes due to remaining shear modulus μ of the half-plane Ω. Taking this fact into account, it is more convenient now for us to use the parameter $\bar{\tau} = \tau\mu_+ = 0.1$; 0.01; 0.001 instead of $\tau_* = \tau\mu$, which depends not

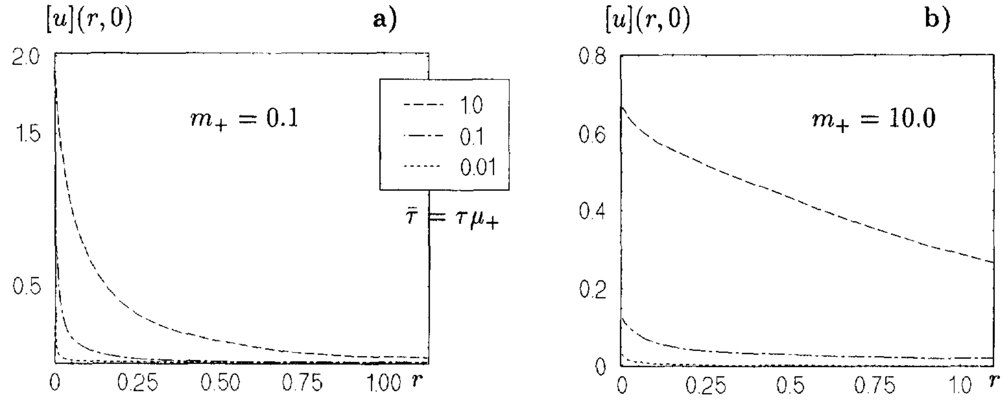

Fig. 7. The graphs of the displacement discontinuity $[u](r,0)$ along the interface Γ_+ in the case $\mu_+ = \mu$ for different values of $\bar{\tau} = \tau_*/m_+$ and the ratio $m_+ = 0.1$; 10.0.

only on the parameter τ, but on the value of μ as well. Graphs in Fig. 7a correspond to ratio $m_+ = \mu/\mu_+ = 0.1$, but those in Fig. 7b correspond to $m_+ = 10.0$. Besides, from these graphs the values of traction along the interface can be also calculated by the relation (1), which is in the form: $\sigma_{x_2z}(r,0) = \tau^{-1}[u](r,0)$.

As one can see, the displacement discontinuity is not equal to zero near the crack tip, as has been mentioned above. The greatest values of $[u](r,0)$ (and the interfacial traction, consequently) arise when the crack terminates in soft material ($m_+ < 1$). However, the function $[u](r,0)$ is rapidly decreasing when $m_+ < 1$ in comparison with the case when the crack terminates in stiff material ($m_+ > 1$). The interfacial crack arises if $[u](0_+,0)$ is greater than some value δ_a, which is a constant of adhesive material. Hence, the risk of an interfacial crack appearing is largest when the crack approaches a soft material from a stiff one. This result completely coincides with that obtained by the theory of adhesion (e.g. Cherepanov (1983)). However, this approach makes it possible to simply investigate the case when the crack terminates in stiff material as well.

As it follows from the Fig. 7 the values of $[u](r,0)$ increase when the magnitude of $\bar{\tau}$ increases, and if $\bar{\tau} \to 0$ then $[u](r,0) \to 0$. To illustrate the last fact in Fig. 8 the values of the jump of the displacement $[u](0_+,0)$ and the normed traction $\mu^{-1}\sigma_{x_2z}(0_+,0)$ are shown as the functions of the parameter τ^*. The graphs are represented in a logarithmic scale for different magnitudes of the ratio $m_+ = \mu/\mu_+$. All other parameters are as above in Fig. 7.

As it can be seen for small $\tau^* < 0.1$, the graphs of the traction and the normed jump of the displacement are straight lines for all presented magnitudes of the parameter m_+. It means that the asymptotics of these values are in the form of

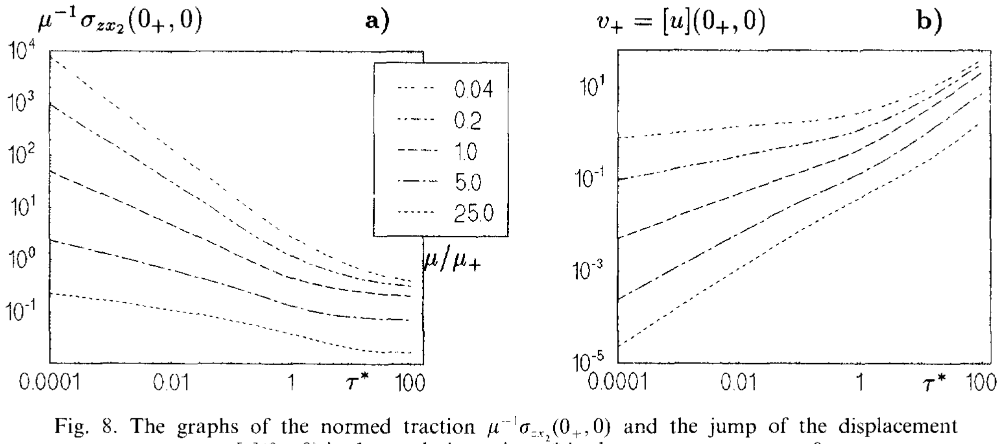

Fig. 8. The graphs of the normed traction $\mu^{-1}\sigma_{zx_2}(0_+,0)$ and the jump of the displacement $v_+ = [u](0_+,0)$ in the crack tip against τ^* in the case $\mu_+ = \mu_-$, $\tau_\pm = 0$.

$$[u](0_+,0) \sim \text{Const}_1(\tau^*)^{\nu}, \quad \sigma_{x_2 z}(0_+,0) \sim \text{Const}_2(\tau^*)^{\nu-1}, \quad \tau^* \to 0, \tag{29}$$

where the exponent ν calculated numerically is connected with the parameter $\omega_1 = \omega_1(\mu/\mu_+)$ correspond to the "ideal" contact along the interface with the accuracy to 0.1%, i.e. $\nu \cong \omega_1$. Note only that the parameters m_+ and τ_* depend on three material parameters μ, μ_+ and τ.

To conclude, note that for the case under consideration ($\tau > 0$), all values of the parameters from the class LW (see (3)) have been found $\gamma_0 = 1, \gamma_\infty = \omega_1$, and the constants $v_\pm$, u_* are presented in (28). Besides, the first relation in the point 3 (3) can be corrected. Namely, it should be $O(r \ln r)$ instead of $O(r^{\gamma_0})$.

6. CONCLUSIONS

As it would be expected, the singularity of the stresses near the crack tip terminating at the bimaterial interface depends on the models of the interface in an essential way.

In the considered Mode III problem for the model of interface in the form: $([u]-r\tau_\pm\sigma)_{|\Gamma_\pm} = 0, [\sigma]_{|\Gamma_\pm} = 0$ (corresponding to the adhesive region in the form of two thin wedges only), the main exponent of the singularity is in the interval $(-1,0)$. It has a value approximating that of the "ideal" bimaterial contact case, if the parameters $\tau^*_\pm < 0.1$. Besides, there is a second exponent in the interval $(-1,0)$, which has the value near zero for $\tau^*_\pm < 0.1$. This singularity cannot be observed in situations when all geometrical and mechanical parameters of the problem are symmetrical with respect to OX_2 axis. The graphs of the singularity exponents ω_1, ω_2 for different values of the parameters are presented in Figs 4, 5.

When the geometry of the adhesive is assumed to be of the general form $([u]-(r\tau_\pm+\tau)\sigma)_{|\Gamma_\pm} = 0, [\sigma]_{|\Gamma_\pm} = 0(\tau,\tau_\pm > 0)$ or the thin layer (where $\tau > 0, \tau_\pm = 0$), the stresses increase in the neighbourhood of the crack tip as $\ln r$ inside the half-plane Ω only, and the greatest value of the stresses $\sigma_{\theta z}$ into the region Ω arises beyond the crack line. However, inside the domains $\Omega_\pm$, the stresses are bounded as well as the traction along the interface. At this time, the jump of the displacements along the interface is not equal to zero near the crack tip. Moreover, for the small magnitudes of the parameter τ^*, the traction and the jump of the displacement near the crack tip are presented in the asymptotic form (29). Thus it is sufficient to calculate these values for a unique small τ^*.

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APPENDIX: SOLVING OF SYSTEM OF FUNCTIONAL EQUATIONS (26)

What remains is to solve the system of functional eqns (26) with condition (9). To this aim, it will be suitable for us to introduce a new unknown function:

$$iC_-(\lambda) = C^*_-(\lambda)+\hat{h}_2(0)\,\mathrm{e}^{-\lambda},\ \mathbf{C}^*(\lambda) = (C_+(\lambda), C^*_-(\lambda))^{\mathrm{T}}, \tag{A1}$$

then $i\hat{C}_-(s) = \hat{C}^*_-(s)+\hat{h}_2(0)\Gamma(1-s)$, $\hat{\mathbf{C}}^*(s) = (\hat{C}_+(s), \hat{C}^*_-(s))^{\mathrm{T}}$, and the vector-functions $\hat{\mathbf{C}}(s)$, $\hat{\mathbf{C}}^*(s)$ will have similar behaviour. Hence, the vector-function $\mathbf{C}^*(\lambda)$ belongs to the space (25).

The system of the eqns (26) defined in the strip $0 < \mathscr{R}s < \omega_1$ is rewritten in the form

$$\tau^*\Phi^{-1}(s)\hat{\mathbf{C}}^*(s-1)+\hat{\mathbf{C}}^*(s) = \hat{H}^*(s), \tag{A2}$$

where

$$H^*(\lambda) = H(\lambda)-\frac{\hat{h}_2(0)}{2\pi i}\int_{-i\infty+\delta}^{i\infty+\delta}[\tau^*(1-s)\Phi^{-1}(s)+\mathbf{I}]\begin{pmatrix}0\\1\end{pmatrix}\Gamma(1-s)\lambda^{s-1}\,\mathrm{d}s,\quad \delta\in(0,\omega_1). \tag{A3}$$

Here $\mathbf{I}$ is the unity matrix. Besides, the known vector-function $\hat{H}^*(s)$ and the unknown vector-function $\hat{\mathbf{C}}^*(s)$ are analytical in the strip $-1 < \mathscr{R}s < \omega_1$ at least, and satisfy the additional conditions

$$\hat{C}^*_-(0) = 0,\quad \hat{h}^*_2(0) = 0. \tag{A4}$$

We shall seek the solutions of the system (A2) with the additional condition (A4) in the form:

$$\hat{\mathbf{C}}^*(s) = R_k^{-1}(s)\hat{D}(s),\quad R_2(s) = \Gamma(s+1)\cos\frac{\pi s}{2}\begin{bmatrix}1 & 0\\ 0 & \operatorname{ctg}\dfrac{\pi s}{2}\end{bmatrix},$$

$$R_1(s) = \begin{bmatrix}1 & 0\\ 0 & \operatorname{ctg}\dfrac{\pi s}{2}\end{bmatrix},\quad R_3(s) = \frac{1}{2}\begin{bmatrix}\Gamma(s+1)\cos\dfrac{\pi s}{2} & \Gamma(s+1)\cos\dfrac{\pi s}{2}\operatorname{ctg}\dfrac{\pi s}{2}\\ 1 & -\operatorname{ctg}\dfrac{\pi s}{2}\end{bmatrix}. \tag{A5}$$

Here and before the values of $k = 1, 2, 3$ correspond to three different cases mentioned above. It is evident that the vector-function $\hat{D}(s)$ has not poles in the points $s = 0$ and $s = -1$ in view to (A4), and the strips of analyticity for vector-functions $\hat{D}(s)$, $\hat{\mathbf{C}}^*(s)$ coincide. Hence, the vector-function $D(\lambda)$ belongs to the spaces

$$D\in L_2^{1,1-\omega_1+\varepsilon,2+\varepsilon_1}(\mathbb{R}_+)\subset L_2^{1,1-\omega_1-\varepsilon,2-\varepsilon}(\mathbb{R}_-)$$

for any small value of $\varepsilon > 0$, and some $\varepsilon_1 > 0$.

Represent matrix-functions $N_k(s) = R_k(s)\Phi^{-1}(s)R_k^{-1}(s-1)$ in the forms

$$N_k(s) = M_1^k(s)+M_2^k(s), \quad M_1^{(3)}(s) = \frac{1}{2\mu t_-[1+m_- - m_-]}\begin{pmatrix} 1+m_+ - m_- & 0 \\ 0 & 2\mu t_+ \end{pmatrix},$$

$$M_1^{(1)}(s) = \frac{1}{(1+m_+)^2 - m_-^2}\begin{pmatrix} 1+m_+ & -m_- \\ -m & 1+m_- \end{pmatrix}, \quad M_1^{(2)}(s) = \frac{1}{t_+^2 - t_-^2}\begin{pmatrix} t_- & t_- \\ t_- & t_+ \end{pmatrix}, \tag{A6}$$

where the matrix-functions $M_2^{(k)}(s)$ hold the condition: $M_2^{(k)}(s) = O(1/s), |\mathscr{F}s| \to \infty$, at least.

Then substituting (A5) in (A2) and applying the inverse Mellin transform

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} [\tau^*(M_1^{(k)}(s)+M_2^{(k)}(s))\hat{D}(s-1)+\hat{D}(s)]\lambda^{s-1}\,\mathrm{d}s = Q_k(\lambda),$$

is obtained where

$$Q_k(\lambda) \equiv \frac{1}{2\pi i}\int_{-i\infty+\delta}^{i\infty+\delta} R_k(s)\hat{H}^*(s)\lambda^{s-1}\,\mathrm{d}s, \quad \delta\in(0,\omega_1).$$

After calculating the respective integrals and taking into account the fact that the matrix-functions $\mathbf{I}+\tau^*\lambda M_1^{(k)}$ are nondegenerated for any $\lambda \geqslant 0$, we obtain systems of singular integral equations in the spaces $L_2^{1,-\omega_1+\varepsilon,1+\varepsilon_1}(\mathbb{R}_+)$ for $\varepsilon, \varepsilon_1 > 0$:

$$(\mathbf{I}+\mathscr{A}_k)Y = Z_k, \quad [\mathscr{A}_k Y](\lambda) = \lambda[\mathbf{I}+\tau^*\lambda M_1^{(k)}]^{-1}\int_0^{\infty}\Psi_k(\lambda,\xi)Y(\xi)\frac{\mathrm{d}\xi}{\lambda}, \quad k = 1,2,3. \tag{A7}$$

Here, we denote by $Y(\lambda) = \lambda D(\lambda)$. Besides, the homogeneous of the degree 0 matrix-function $\Psi_k(\lambda,\xi) = \Psi_k(\lambda/\xi, 1)$ and the vector-function $Z_k(\lambda)$ are defined by relations:

$$\Psi_k(t,1) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\tau^* M_2^{(k)}(s)t^s\,\mathrm{d}s, \quad Z_k(\lambda) = \lambda[\mathbf{I}+\tau^*\lambda M_1^{(k)}]^{-1}Q_k(\lambda), \tag{A8}$$

Note, that the integral operators $\mathscr{A}_k$ have fixed singularities in zero and at infinity points. Based on the results from the paper Mishuris and Olesiak (1995), it can be shown that the operators $\mathscr{A}_k$: $L_2^{p,\alpha,\beta}(\mathbb{R}_+) \to L_2^{p,\alpha,\beta}(\mathbb{R}_+)$ $(-\omega_1 < \alpha < \beta < 1+\omega_1, 1 \leqslant p < \infty)$ are bounded, and the following theorem can be proved:

Theorem

The system of the singular integral equation (A7) has a unique solution in the space $L^{p,\alpha,\beta}(\mathbb{R}_+)$ $(1 \leqslant p < \infty, 1-\omega_1 < \beta < 1+\omega_1, -\omega_1 < \alpha \leqslant \beta)$, which can be calculated by projectional methods.

We do not present here the proof of the theorem. Note only that the main aim is to show that the operators $\mathbf{I}+\mathscr{A}_k$ in the space $L^{p,\alpha,\beta}(\mathbb{R}_+)$ $(1 \leqslant p < \infty, 1-\omega_1 < \beta < 1+\omega_1, -\omega_1 < \alpha \leqslant \beta)$ are isometrically equivalent to some pair integral of the operator Wiener-Hopf type accurate to a compact operator. The symbols of those operators are of the form

$$\mathrm{Symb}[\mathbf{I}+\mathscr{A}_k]|_{L^{p,\alpha,\beta}(\mathbb{R})}(t,\zeta) = \frac{1+\zeta}{2}\mathbf{I}+\frac{1-\zeta}{2}[M_1^{(k)}]^{-1}N_k(\beta-1-it), \quad t\in\overline{\mathbb{R}}, \quad \zeta = \pm 1.$$

Then it remains to note that $N_k(-it)$ are the definite matrix-functions, and the even functions $\det N_1(-it) = \mathrm{cth}^2 t\det\Phi^{-1}(-it)$, $\det N_2(-it) = t^2\mathrm{cth}^4 t\det\Phi^{-1}(-it)$, $\det N_3(-it) = t\mathrm{cth}^3 t\det\Phi^{-1}(-it)$ have not any zeros for $t\in\mathbb{R}$. Consequently, the index and the part indices of the operators $\mathbf{I}+\mathscr{A}_k$ in this space are equal to 0.

When the solution $Y(\lambda)$ of the eqn (A8) is found, we can then calculate the vector-function $\mathbf{C}^*(\lambda)$ (see (A5)) as

$$\mathbf{C}^*(\lambda) = [\mathscr{R}_k Y](\lambda), \quad \mathscr{R}_1 = \begin{pmatrix} \frac{1}{\lambda}\mathbf{I} & 0 \\ 0 & \mathscr{B}_1 \end{pmatrix}, \quad \mathscr{R}_2 = \begin{pmatrix} \mathscr{B}_2 & 0 \\ 0 & \mathscr{B}_3 \end{pmatrix}, \quad \mathscr{R}_3 = \begin{pmatrix} \mathscr{B}_2 & \frac{1}{\lambda}\mathbf{I} \\ \mathscr{B}_3 & -\mathscr{B}_1 \end{pmatrix},$$

$$[\mathscr{B}_1 v](\lambda) = \frac{2}{\pi}\int_0^{\infty}\frac{v(\xi)\,\mathrm{d}\xi}{\xi^2-\lambda^2}, \quad [\mathscr{B}_2 v](\lambda) = \frac{2}{\pi}\int_0^{\infty}\sin(\lambda/\xi)v(\xi)\frac{\mathrm{d}\xi}{\lambda\xi},$$

$$[\mathscr{B}_3 v](\lambda) = \frac{4}{\pi^2}\int_0^{\infty}[\mathrm{Si}(\lambda/\xi)\cos(\lambda/\xi)+\mathrm{ci}(\lambda/\xi)\sin(\lambda/\xi)]v(\xi)\frac{\mathrm{d}\xi}{\lambda\xi}, \tag{A9}$$

where $\mathbf{I}$ is the unity operator and singular integral operators $\mathscr{B}_k : L^{p,\alpha,\beta}(\mathbb{R}_+) \to L^{p,\alpha+1,2-\varepsilon}(\mathbb{R}_+)$ are bounded for any $-\omega_1 < \alpha < \beta < 1+\omega_1$, $1 \leqslant p < \infty$, $\varepsilon > 0$. Hence, all *a priori* estimates for the vector-function $\mathbf{C}(\lambda)$ (25) have been justified. What remains is to calculate the values of the respective parameters $\hat{C}_+(-1), e_-$ from the asymptotics of the problem solution (28)

$$\begin{pmatrix} \hat{C}_+(-1) \\ e \end{pmatrix} = A_k \hat{Y}(0), \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -\dfrac{2}{\pi} \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \dfrac{2}{\pi} & 0 \\ 0 & -\dfrac{4}{\pi^2} \end{bmatrix}, \quad A_3 = \begin{bmatrix} \dfrac{2}{\pi} & 1 \\ -\dfrac{4}{\pi^2} & \dfrac{2}{\pi} \end{bmatrix}.$$